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Advances in the Study of Smoothing Methods for Correlation Matrices in Confirmatory Factor Analysis

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ABSTRACT

Confirmatory factor analysis (CFA) using polychoric correlations has become standard in psychometric and item analyses. Nevertheless, sparse data can lead to non-positive definite (NPD) polychoric correlation matrices, posing notable challenges. Smoothing algorithms to address this issue can play an important role in eliminating noise and enhancing signal quality. In the present article, simulation studies were conducted to compare the eigenvalue-based smoothing methods, semidefinite dual approaches, and Sylvester's criterion smoothing methods. They all aim to transform NPD matrices into positive definite ones, but differ in technique. To address the limitations of these methods, a new smoothing algorithm is proposed at the end.

KEYWORDS

CFA; non-positive definite polychoric correlation; smoothing algorithm

1. Introduction

In Structural Equation Modeling (SEM) and Confirmatory Factor Analysis (CFA), relationships between latent constructs and observed indicators are typically modeled using correlation or covariance matrices (Kline, 2023). While Pearson correlations are suitable for continuous data, they can underestimate associations in ordinal data due to the unequal spacing between categories (Olsson, 1979; Robitzsch, 2020). Polychoric correlations address this limitation by assuming an underlying bivariate normal distribution (Robitzsch, 2020), making them the standard choice for analyzing ordinal data in psychometrics (Mueller & Hancock, 2015). However, estimating polychoric correlations can be computationally demanding, as it involves multidimensional integrals. A computational workaround implies estimating each correlation for pairs of variables, but the process often yields non-positive definite (NPD) matrices (i.e., matrices with one or more negative eigenvalues), especially when data are sparse, which can complicate model estimation (Ekström, 2011).

Smoothing algorithms have emerged as essential tools for converting NPD polychoric matrices into positive definite forms. The application and evaluation of smoothing algorithms in Exploratory Factor Analysis (EFA) and Principal Component Analysis (PCA) have been extensively studied over the past few decades. Hayashi and Marcoulides (2006) offer an in-depth overview of this issue in the context of exploratory factor analysis, with a more recent discussion by Marôco (2024).

Debelak and Tran (2013) conducted one of the foundational studies on the accuracy of parallel analysis when applied to potentially non-positive definite (NPD) tetrachoric correlation matrices. Their work focused on dimensionality recovery using three distinct smoothing methods: (a) the Higham alternating-projections algorithm (Higham, 2002), (b) the Bentler and Yuan algorithm (Bentler & Yuan, 2011), and (c) the Knol and Berger algorithm (Knol & Berger, 1991). Their study highlighted the importance of smoothing for improving dimensionality assessment in tetrachoric correlation matrices. In a subsequent study, Debelak and Tran (2016) extended their analysis to polychoric correlation matrices, evaluating dimensionality recovery across a broader range of major common factors. Building on this work, Kracht and Waller (2022) replicated these assessments of dimensionality in one- and two-dimensional common factor models, emphasizing the importance of ensuring that matrices are PD before analyzing them.

Lorenzo-Seva and Ferrando (2020) investigated the causes, consequences, and potential solutions for non-positive definite (NPD) polychoric correlation matrices in Exploratory Item Factor Analysis. They evaluated five smoothing methods: (a) least-squares smoothing (Knol & ten Berge, 1989), (b) linear smoothing via a ridge penalty (Jöreskog & Sörbom, 1981), (c) non-linear smoothing (Devlin et al., 1975, 1981), (d) the Bentler–Yuan algorithm (Bentler & Yuan, 2011), and (e) a novel "sweet smoothing" algorithm proposed by Lorenzo–Seva and Ferrando themselves. Their study included simulations comparing the performance of the Bentler–Yuan and sweet smoothing algorithms, highlighting the relative merits of each.

Further contributions were made by Nilforooshan (2020), who examined three additional methods: (a) an iterative weighting procedure (Jorjani et al., 2003), (b) the unweighted bending procedure (Schaeffer, 2014), and (c) a method proposed by Bock et al. (1988). Nilforooshan evaluated the differences between pre- and post-smoothing results across covariance matrices, correlation matrices, and ill-conditioned matrices. This study broadened understanding of the practical effects of smoothing, clarified when such approaches are necessary, and compared the relative advantages of each method.

Despite advances in exploratory contexts, research on smoothing algorithms for confirmatory factor analysis remains underdeveloped. The present study addresses this critical gap through systematic simulations evaluating (a) convergence rates, (b) parameter estimation accuracy, and (c) model fit indices in CFA applications. To help resolve the pervasive issue of non-positive definite (NPD) matrices, we propose a novel smoothing procedure inspired by Lorenzo-Seva and Ferrando (2020).

1.1. Matrix Smoothing Algorithms

In this article, we compared six approaches proposed by Bock et al. (1988), Knol and Berger (1991), Schaeffer (2014), Higham (2002), and Bentler and Yuan (2011) (see Table 1). Table 1 presents an overview of these methods, organized by author(s), methodological type, algorithmic details, and R packages. Based on the underlying principles, the methods are categorized as eigenvalue-based, semidefinite dual, or Sylvester's criterion approaches. Our novel method (inspired by Lorenzo-Seva & Ferrando, 2020) relies on Sylvester's criterion of PD and is categorized as such. This novel smoothing method conducts a careful iterative process of Higham's nearest correlation approach across the principal minors. It ensures that the correlation matrix remains as close as possible to being positive definite while extracting the appropriate part of the matrix at each step (see details in Table 1 of the last row). We also included this approach in simulation scenarios for evaluation to assess whether it could achieve our intended goals.

2. Methodology

A Monte Carlo simulation was implemented to compare the six smoothing approaches, including our new approach, using the R packages and functions presented in Table 1. CFA was conducted on the smoothed correlation matrix using the six approaches described above.

2.1. Data Generation

We generated simulated datasets in lavaan (Rosseel, 2012) by implementing three factor structures: (a) a correlated two-factor model with six binary indicators per factor (Figure 1), (b) a one-factor model representing core applications of item response theory (IRT; Figure 2; Wirth & Edwards, 2007), and (c) a bifactor model in which two specific factors were each measured by six binary indicators, while a general factor was measured by all twelve indicators. We allowed the specific factors to correlate with each other but specified them as orthogonal to the general factor (Figure 3). This design facilitated the evaluation of smoothing algorithm performance across commonly employed but structurally distinct SEM configurations.

To systematically evaluate method performance, we employed a fully crossed 3×3 factorial design with 1,000 replications per condition ($R = 1000$). Factor loadings and latent correlations were uniformly sampled across three ranges: low (0 to 0.3), medium (0.3 to 0.6), and high (0.6 to 0.9), ensuring a comprehensive assessment of parameter recovery under varying population conditions. Albeit uncommon, this simulation method mimics the logic behind the design of experiments with a random factor to ensure conclusions generalize across the range of potential parameter estimates (Li & Zumbo, 2009; Shear & Zumbo, 2013).

Data sparseness is a known contributor to estimation imprecision and the generation of non-positive definite correlation matrices. We controlled the degree of sparseness through threshold manipulation. Specifically, we simulated binary responses by alternating thresholds of 1.25 and −1.75 across items (e.g., item 1 had a threshold of 1.25, item 2 had a threshold of −1.75, item 3 had a threshold of 1.25, and so forth). This intentional use of extreme thresholds induced sparseness, thereby increasing the likelihood of non-positive definiteness. Five sample sizes (n) levels were examined: 100, 200, 300, 500, 1000 cases (see Li, 2016a, 2016b for a similar design).

After generating the data, we conducted a series of analytic steps to evaluate model performance. First, we performed confirmatory factor analysis (CFA) using the true population covariance matrix to derive the asymptotic matrix (Γ) for standard error computation. Next, we calculated the polychoric correlation matrix from the simulated data and applied five distinct smoothing methods to address potential non-positive definiteness. Finally, we fitted the correct factor model to each smoothed matrix to assess convergence rates, parameter recovery, and overall model fit.

2.2. Simulation Outcomes

2.2.1. Convergence

Since small samples are known to induce issues of model non-convergence in SEM, the proportion (M) of non-convergent models was recorded.

2.2.2. Bias

The raw bias of each simulation condition, $B(\theta_c)$ was calculated as:

$$B(\theta_c) = (RN)^{-1}\Sigma_{r=1}^{R}\Sigma_{i=1}^{N}(\hat{\theta}_{cri} - \theta_{cri}), \quad (1)$$

where $\hat{\theta}_{cri}$ is the i^{th} parameter estimate for replication r in condition c, θ_{cri} is the corresponding true population parameter. Here, $R = 1000$ denotes the total number of replications, and N represents the total number of estimated parameters. This formula computes the average difference between estimated and true parameter values across all parameters and replications. We systematically compared

Table 1. Matrix of smoothing algorithms.

Author(s)	Type	Algorithm	R function
Bock et al. (1988)	Eigenvalue-based	Step 1: Perform eigen-decomposition of the matrix: $A = Q\Lambda Q^{-1}$, where Q is the matrix of eigenvectors and Λ is the diagonal matrix of eigenvalues. Step 2: Replace each eigenvalue λ_i smaller than a threshold c with 100c (where c is a small constant). Step 3: Rescale the sum of the positive eigenvalues to match the number of items, producing diagonal matrix Λ'. Step 4: Compute the new matrix using the updated eigenvalues: $A' = Q\Lambda' Q^{-1}$. Step 5: Rescale the resulting matrix to a correlation matrix using cov2cor.	cor.smooth(), 'psych' package (Revelle, version 2.3.12)
Knol and Berger (1991)		Step 1: Perform eigen-decomposition of the matrix: $A = Q\Lambda Q^{-1}$. Step 2: Construct new eigenvalue matrix: $\Lambda' = \text{diag}(\max(\lambda_i, 0))$ Step 3: Compute the new matrix using the updated eigenvalues: $A' = Q\Lambda' Q^{-1}$. Step 4: Rescale the resulting matrix to a correlation matrix using cov2cor.	smoothKB(), 'fungible' package (Waller et al., 2023, version 2.4.4)
Schaeffer (2014)		Step 1: Perform eigen-decomposition of the matrix: $A = Q\Lambda Q^{-1}$. Step 2: For each negative eigenvalue λ_i, compute a small positive value using the formula: $\rho(s-\lambda_i)^2/(100s^2+1)$, where: a. ρ is the smallest positive eigenvalue, b. $s = 2\sum_{i=1}^{m}\lambda_i$ is twice the sum of all negative eigenvalues, c. m is the number of negative eigenvalues. Step 3: Replace negative λ_i with small positive values from step 2 in descending order.	bend(),'mbend' (Nilforooshan, version 1.3.1)
Higham (2002)	semidefinite dual	Step 1: Project the matrix A onto the set of symmetric positive semidefinite matrices S: $(W^{-1} \circ W^{-1}) = \text{diag}(A\text{-}I)$; Step 2: Project A onto the set of matrices with unit diagonals U: $Ps(A) = W^{-1/2}((W^{1/2}AW^{1/2})_+)W^{-1/2}$ where $(\cdot)_+$ retains the positive semidefinite part of the matrix. Step 3: Use the Frobenius norm to measure the closeness between smoothed a) and non-smoothed, b) matrices (Havel, 2002; Golub & Van Loan, 2013): $\lVert A\text{-}B \rVert_F^2 = \sum_i \sum_j (a_{ij} - b_{ij})^2$	nearcorr(), 'sfsmisc' package (Maechler, 2024,Version 1.1-19)
Bentler and Yuan (2011)	Sylvester's criterion	Step 1: Extract all potential factors from the common factor space and identify variables with communalities >1. Step 2: Reduce specific correlation estimates by a context-sensitive factor k. Step 3: Progressively decrease $\left(\frac{0.0010}{N^{1/2}}\right)$; until: a. The matrix becomes positive definite, or b. $k \leq 0$	smoothBY(), 'fungible' package (Waller et al., 2023, version 2.4.4)
Proposed new approach (inspired by Lorenzo-Seva & Ferrando)		Step 1: Set the number of factors to extract $r = 1$. Step 2: Extract r-factors from A, and check for non-positive NPD. Step 3: If no NPD is observed, increase r by 1 and repeat step 2. Otherwise: a. Extract the first $r * r$ submatrix from A. b. Apply Higham's nearest correlation smoothing to this submatrix. c. Replace the $r * r$ block in A with the smoothed submatrix to create A'. Step 4: Increase r by 1 and repeat steps 2-3 until the entire matrix is reconstructed and smoothed.	

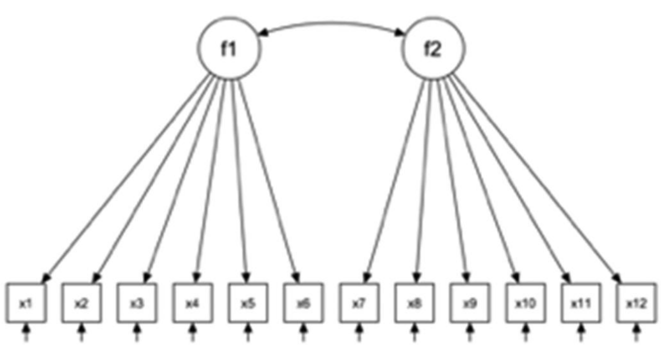

Figure 1. Two-factor model with 12 binary items. Factor 1 is associated with 6 items, ×1 through ×6, while factor 2 is linked to items, ×7 through ×12. Factor 1 and factor 2 are correlated.

the magnitude of these bias estimates across all simulation conditions and smoothing methods for both (a) factor loading estimates and (b) latent factor correlations.

2.2.3. Type I error rate

Since the correct model is being fitted to the data, the p-value associated with the χ^2 test of fit will be used to evaluate the empirical Type I error rate. This is measured as the proportion of p-values less than .05 across R replications.

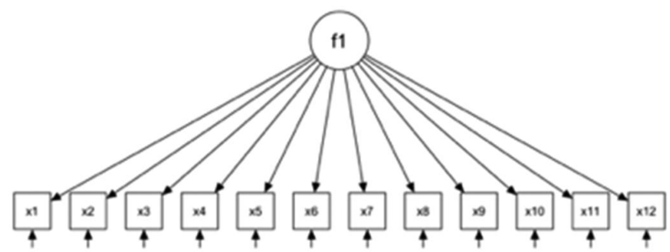

Figure 2. One-factor model with 12 binary items. Factor 1 is associated with all 12 items, ×1 through ×12.

2.2.4. Approximate model fit

Robust Comparative Fit Index (CFI) and Robust Root Mean Square Error of Approximation (RMSEA) are approximate fit indices less sensitive to sample size. The CFI compares the specified model with a baseline model, assuming no relationships among the variables. Values closer to 1 indicate a better fit, with a threshold of 0.95 or above generally indicating a good model fit (Hu & Bentler, 1999). Values of RMSEA less than 0.05 indicate a close fit (Browne &

Cudeck, 1992). Since the correct model is being fitted to the data, we would expect CFI = 1.0 and RMSEA = 0.0. We will monitor the proportion of CFI and RMSEA values different from these values.

3. Results

Because the results and trends were similar across the three models, we selected the two-factor model as a representative example to describe in detail. Results for the other two models are provided in the supplementary material. For the two-factor model, we present the results of the Monte Carlo study as follows. First, we report the average bias for factor loadings and latent factor correlations, along with their 95% confidence intervals. Second, we summarize the Type I error rates, RMSEA, and CFI values according to the different simulation conditions.

3.1. Convergence

The simulation results revealed differences in convergence rates across methods (see Figure 4). In the first panel of this figure, results were grouped by model type. Specifically, LL, ML, and HL referred to low, moderate, and high factor loadings, respectively, while LC, MC, and HC represented low, moderate, and high factor correlations (low: 0 to 0.3, moderate: 0.3 to 0.6, high: 0.6 to 0.9). As shown in the figure, the Bentler and Yuan smoothing algorithm demonstrated particularly poor performance, failing to converge in the majority of replications, whereas the other methods maintained substantially higher convergence rates. Convergence for the Bentler and Yuan (BY) method improved as factor loadings and correlations increased from low to moderate levels. However, this trend reversed at higher levels of loadings and correlations, where convergence rates declined. In contrast, the other five algorithms showed a consistent improvement in convergence rates across the full range of loading and correlation conditions.

The second panel in Figure 4 presents results grouped by sample size (i.e., 100, 200, 300, 500, and 1,000). This panel illustrates how variations in sample size further differentiated method performance. The Bentler and Yuan (BY) algorithm exhibited declining convergence rates as the sample size increased from small ($n = 100$) to moderate levels ($n = 200$ or $n = 300$), whereas the competing methods showed improved performance. However, with larger samples ($n = 500$ or $n = 1{,}000$), the Bentler and Yuan method demonstrated a dramatic recovery (from M = 10% to M > 80% convergence), while the other methods experienced slight declines. Across all conditions, the new approach and Higham's nearest correlation method (HI) proved to be the most reliable, consistently maintaining high convergence rates. The worsening convergence observed for some methods in specific models may reflect increasing model instability or structural incompatibility with larger datasets.

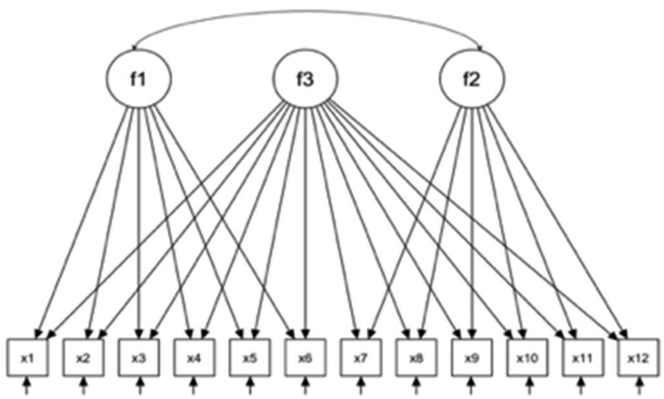

Figure 3. Bi-factor model with 12 binary items. Factor 1 is associated with six items, ×1 through ×6, while factor 2 is linked to item ×7 through ×12. Factor 1 and factor 2 are correlated. Factor 3 is associated with all 12 items, ×1 through ×12.

3.2. Factor Loading Estimation

3.2.1. Two-Factor Model

Due to convergence issues with the Bentler and Yuan method, we excluded it from further analysis. We evaluated the five remaining smoothing methods by calculating their mean estimation bias in a two-factor confirmatory model. Figure 5 presents these mean differences with 95% confidence intervals (±1.96 standard errors), displaying results

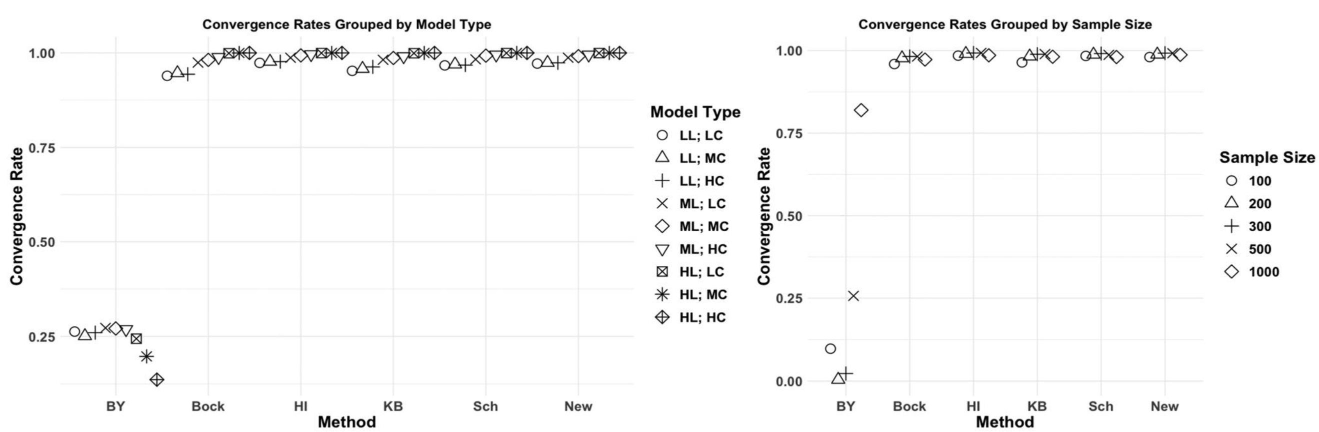

Figure 4. Comparison of convergence rates for six methods in two-factor model. Panel 1: Comparison of convergence rates for six methods grouped by model Type. Panel 2: Comparison of convergence rates for 6 methods grouped by sample size. The figure demonstrates convergence rates for Bentler and Yuan (2011) (by), Higham (2002) (HI), Bock et al. (1988) (Bock), Knol and Berger (1991) (KB), Schaeffer (2014) (sch), and the new approach (new).

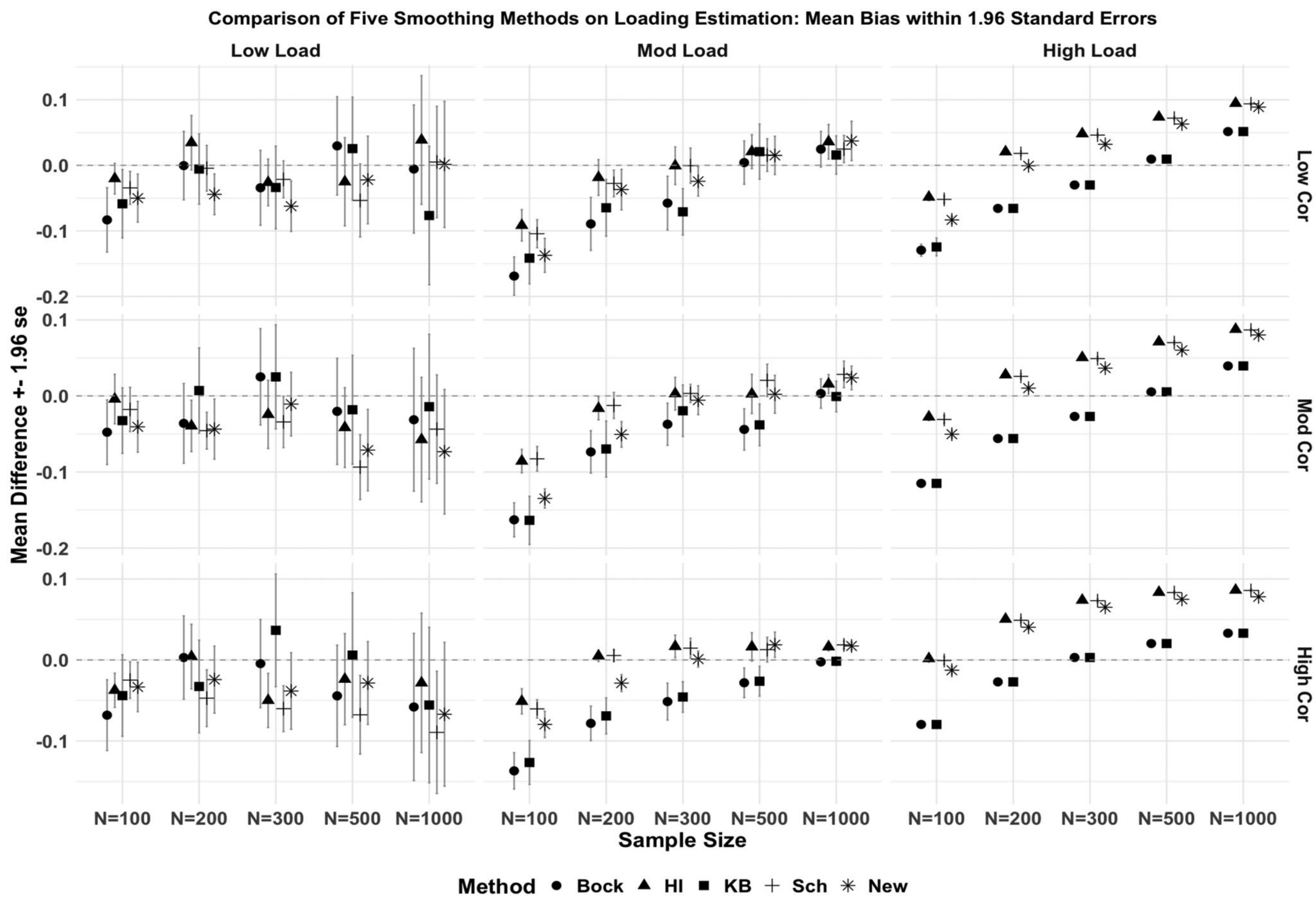

Figure 5. Comparison of loading estimation for five methods in two-factor model: Mean bias within 1.96 standard errors. The figure demonstrates loading estimates for Higham (2002) (HI), Bock et al. (1988) (Bock), Knol and Berger (1991) (KB), Schaeffer (2014) (sch), and the new approach (new).

across all methods, sample sizes, and three levels of loading conditions (low: 0 to 0.3, moderate: 0.3 to 0.6, and high: 0.6 to 0.9) and latent variable correlations. This presentation provides a comprehensive assessment of parameter recovery accuracy.

Analyses of the moderate and high loading conditions revealed systematic performance differences across methods. Higham's method (HI) and Schaeffer's method (Sch) consistently produced the highest estimates, followed by the proposed new method, whereas Bock et al.'s (Bock) method and the Knol and Berger (KB) method yielded the lowest estimates. At the smallest sample size ($N = 100$), all methods exhibited downward bias, although the estimates from HI and Sch were closest to the true values. This pattern shifted with increasing sample size: the new method demonstrated optimal accuracy at moderate sample sizes ($N = 200$ or $N = 300$), while the Bock and KB methods performed best at larger sample sizes ($N = 500$ or $N = 1{,}000$). Two notable trends emerged across these conditions: (a) estimation accuracy improved with increasing sample size under moderate loadings at all correlation levels, and (b) variance decreased monotonically as loading conditions progressed from low to high. The performance hierarchy (HI/Sch > new method > Bock/KB) remained remarkably stable across all variations.

Under low loading conditions, we observed a counterintuitive phenomenon in which variance increased with larger sample sizes, contradicting standard statistical expectations. Although all methods exhibited variable performance across conditions, Higham's method (HI) demonstrated superior stability and accuracy in most cases. The proposed new method consistently ranked second in performance, followed closely by the Knol and Berger (KB) method in terms of estimation accuracy.

3.3. Factor Correlation Estimation

3.3.1. Two-factor Model

Figure 6 illustrates the mean differences between estimated and true correlations, along with 95% confidence intervals (±1.96 standard errors), comparing results across five smoothing methods, varying sample sizes, three levels of loading conditions, and three levels of latent variable correlations. Variance consistently decreased as loading conditions progressed from low to high across all correlation levels, regardless of sample size or estimation method. The effects of sample size varied by loading condition: variance systematically decreased as sample size increased under moderate and high loadings, but remained relatively stable under low-loading conditions.

Regarding method performance, the proposed new method produced systematically higher estimates than both Higham's method (HI) and Schaeffer's method (Sch) under high-loading conditions, although all three approaches

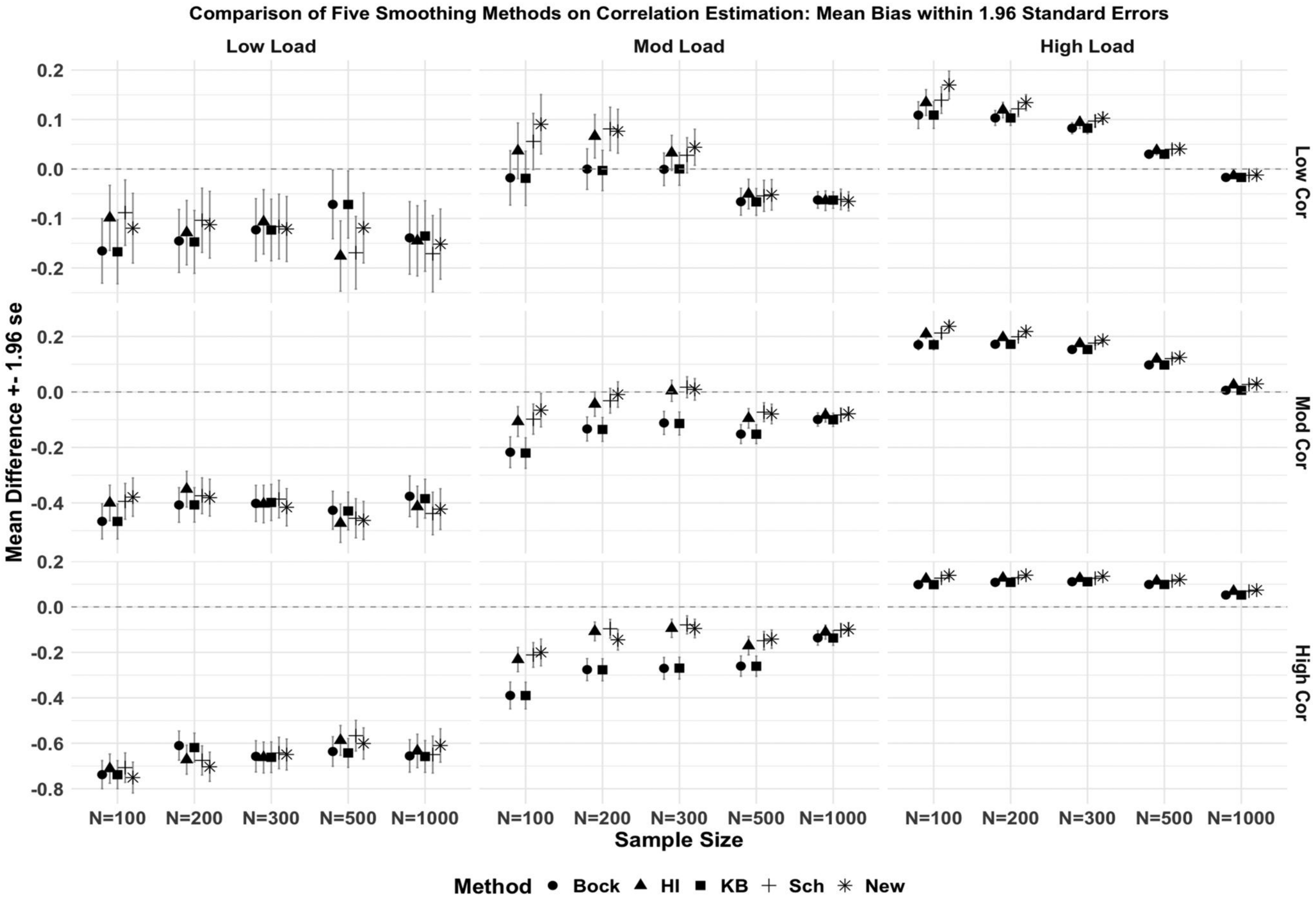

Figure 6. Comparison of correlation estimation for five methods in two-factor model: Mean bias within 1.96 standard errors. The figure demonstrates correlation estimates for Higham (2002) (HI), Bock et al. (1988) (Bock), Knol and Berger (1991) (KB), Schaeffer (2014) (sch), and the new approach (new).

Table 2. Two-factor model fit of moderate loading and moderate correlation with criteria: Robust CFI > 0.95, Robust RMSEA < 0.05, and *P*-value < 0.05. The table demonstrates the fit assessment for the Higham (2002) (HI), Bock et al. (1988) (Bock), Knol and Berger (1991) (KB), Schaeffer (2014) (sch), and the new approach (new) over 1000 replications. It included the expected value and proportions of robust CFA, robust RMSEA, and *P*-value. The simulation condition here was $\lambda = uniform(0.3, 0.6)$ and $\rho = uniform(0.3, 0.6)$.

	Sample Size	HI	Bock	KB	Sch	New
Robust RMSEA (%)	100	0.23	1.87	1.87	0.23	0.23
Robust CFI (%)		0.68	0.93	0.94	0.56	0.34
Type I error (%)		75.6	46.8	46.7	76.4	72.9
Robust RMSEA (%)	200	1.12	4.31	4.21	1.12	2.35
Robust CFI (%)		0.34	0.41	0.53	0.41	0.61
Type I error (%)		92.8	69.1	69.1	93.1	88.0
Robust RMSEA (%)	300	3.94	10.85	10.85	3.83	6.48
Robust CFI (%)		0.61	0.40	0.31	0.50	0.61
Type I error (%)		96.6	86.6	86.6	96.8	93.9
Robust RMSEA (%)	500	15.95	33.81	33.92	16.04	22.19
Robust CFI (%)		0.20	0.10	0.10	0.20	0.20
Type I error (%)		97.1	95.7	95.7	97.2	96.8
Robust RMSEA (%)	1000	73.06	79.49	79.51	73.49	74.46
Robust CFI (%)		3.57	3.18	3.18	3.99	3.78
Type I error (%)		94.2	93.8	93.9	93.7	94.2

yielded larger estimates than either the Bock or Knol and Berger (KB) methods. Despite exhibiting upward bias in these conditions, the Bock and KB methods provided the most accurate parameter estimates. In contrast, low-loading conditions produced consistent downward bias across all methods. For moderate loadings, performance varied by correlation level: the Bock and KB methods excelled at low correlations, the new approach outperformed others at moderate correlations in almost all scenarios, and at high correlations, the new method generally dominated, except at a sample size of $N = 200$, where HI and Sch performed slightly better.

3.4. Fit Assessment

3.4.1. Two-Factor Model

Because the models exhibited similar performance in terms of model fit, we combined the results for the condition with moderate loadings and moderate correlations in Table 2, as this scenario may be the most common in real-world applications. As shown in Table 2, none of the five methods demonstrated a particularly strong fit for the two-factor model. Among the methods, the Bock and Knol and Berger (KB) approaches, both belonging to the eigenvalue-based family, performed similarly and produced the best fit statistics compared to the others. These two methods consistently yielded the highest number of cases in which Robust CFI and RMSEA met the fit criteria, while exhibiting fewer cases with a p-value smaller than 0.05, despite noticeable inflation in Type I error rates. In contrast, Higham's (HI) and Schaeffer's (Sch) methods also behaved similarly but produced the poorest model fit indices, with the weakest performance on Robust CFI, RMSEA, and Type I error rates. The new method fell between the two groups, performing moderately across all metrics without clear superiority or

Table 3. Pros and cons for each smoothing algorithm.

Algorithm	Recommend (+)	Not Recommend (−)
Bock et al. (1988)	Structural Detection/Model Fit Correlation Estimations	Loading Estimation
Knol and Berger (1991)	Structural Detection/Model Fit Correlation Estimations	Loading Estimation
Higham (2002)	Loading Estimation High convergency rate	Structural Detection/ Model Fit Correlation Estimations
Bentler and Yuan (2011)	Correlation Estimations Structural Detection/ Model Fit	Loading Estimations Convergency Issues
Schaeffer (2014)	Loading Estimation	Structural Detection/ Model Fit Correlation Estimations
Proposed New Approach (inspired by Lorenzo-Seva and Ferrando)	Small Sample Size High convergency rate	No apparent drawback

inferiority. These results underscore notable differences among the methods, with the Bock and KB approaches demonstrating relatively better performance in meeting model fit criteria despite certain limitations.

As sample size increased, the performance of Robust RMSEA improved, with a greater number of cases meeting the criterion of Robust RMSEA < 0.05 across all five methods. In contrast, Robust CFI values and Type I error rates deteriorated as sample size increased, particularly up to $N = 500$. However, slight improvements in these indices were observed when the sample size reached $N = 1{,}000$. At larger sample sizes, the smoothing approaches appeared to distort additional information from the matrix, leading to an overall decline in model fit indices' performance.

4. Conclusion

This study evaluated methods for smoothing non-positive definite correlation matrices through theoretical analysis and Monte Carlo simulations. We compared approaches proposed by Bock et al. (1988), Knol and Berger (1991), Higham (2002), Bentler and Yuan (2011), Schaeffer (2014), and a proposed new method, identifying the optimal conditions for each. Table 3 summarizes key model and estimation features for which the different smoothing methodologies are recommended or not recommended.

The Bentler and Yuan method encounters convergence difficulties due to its non-constructive treatment of matrix D. While Theorem 1 proves D's existence, it provides no explicit construction method, relying instead on heuristic determination of an arbitrary constant k. This results in multiple potential algorithm variants with differing outcomes. Preliminary evidence suggests CSDP (a semidefinite programming approach similar to Higham's method) could construct D, though additional research is needed to verify this approach.

Two eigenvalue-based methods (Bock et al., 1988; Knol & Berger, 1991) showed superior model fit and correlation estimation while preserving proportional relationships among correlations. In contrast, Higham's Higham (2002) and Schaeffer's Schaeffer (2014) methods demonstrated stronger loading estimation capabilities, revealing a methodological trade-off between structural detection and factor loading accuracy. Notably, correlation estimation ability covaried with model fit, suggesting shared psychometric properties. The Bentler and Yuan (2011) method, despite convergence challenges, achieved excellent results post-convergence by selectively modifying only problematic correlations while preserving others.

The newly proposed method in this study successfully integrates these complementary strengths while addressing their respective limitations. It maintains high convergence rates comparable to Higham's method while achieving balanced performance across model fit, correlation estimation, and loading accuracy.

5. Limitations and Future Direction

Robust corrections within structural equation modeling (SEM) rely on the weight matrix of asymptotic covariances of the model, Γ. In this simulation study, we observed that the smoothing algorithm alters the original polychoric correlation matrix, from which the weight matrix is derived. As a result, the weight matrix no longer corresponds to the model-implied covariance matrix, which may explain the observed poor fit performance. Future research should investigate how the smoothing algorithm affects the weight matrix and its implications for model fit.

Furthermore, the impact of these methods on standard error estimation remains unknown. The Bentler and Yuan method warrants further investigation, especially regarding its treatment of matrix *D*. Understanding these aspects could provide valuable insights into improving the robustness and applicability of these methods.

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